

Hydrodynamic accretion onto rapidly rotating Kerr black hole

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Abstract

Bondi type hydrodynamic accretion of the surrounding matter onto Kerr black hole with an arbitrary rotational parameter is considered. The effects of viscosity, thermal conductivity and interaction with radiation field are neglected. The black hole is supposed to be at rest with respect to matter at infinity. The flow is adiabatic and has no angular momentum. The fact that usually in astrophysics substance far from the black hole has nonrelativistic temperature introduces small parameter to the problem and allows to search for the solution as a perturbation to the accretion of a cold, that is dust-like, matter. However, far from the black hole on the scales of order of the radius of the sonic surface the expansion must be performed with respect to Bondi spherically symmetrical solution for the accretion on a Newtonian gravitating centre. The equations thus obtained are solved analytically. The conditions of the regularity of the solution at the sonic surface and at infinity allow to specify unique solution, to find the shape of the sonic surface and to determine the corrections to Bondi accretion rate.

Key words: accretion – black holes: hydrodynamics – black hole

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1 Introduction

The problem of the matter accretion onto gravitating objects is the classical problem of astrophysics (Zeldovich & Novikov 1967; Shapiro & Teukolsky 1983). The accretion is widely engaged for the explanation of various kinds of astrophysical phenomena. It is believed that this process is basic to understanding the nature of active galactic nuclei, quasars, galactic X-ray sources, the formation of extragalactic jets and jets from some galactic objects of stellar masses (i.e., Begelman, Blandford & Rees 1984; Camenzind 1990; Lipunov 1992). Usually, when considering accretion with astrophysical implications, the authors either worked in the frame of a disk accretion model or considered spherically symmetrical case, which enabled them without excessive details to estimate the accretion rate and the emerging radiation due to the heating of infalling matter (Shapiro & Teukolsky 1983). Bondi (1952) was the first who considered hydrodynamic isentropic stationary flow for polytropic equation of state and showed that, being subsonic at infinity, flow must cross the sonic surface and become supersonic near the black hole, the accretion rate being determined by the requirement of smoothness of the solution at the sonic surface.

Present paper investigates the adiabatic stationary accretion onto a rotating black hole. Let us first note that the luminosity of the infalling matter itself is much less than the Eddington limit in quasispherical case (Shapiro & Teukolsky 1983), so, despite that our approach does not involve interaction with photons, it can be applied for the accretion by an isolated black hole embedded in interstellar gas or molecular cloud. Probably Shapiro (1974) was the first who considered accretion onto a Kerr black hole. He noticed that in the case of zero temperature the situation is similar to the accretion of dust, so the particles would follow geodesic lines in Kerr space-time and flow lines would simply coincide with them. As it can be verified directly from the equations for the test particle motion in Kerr geometry (e.g., Novikov & Frolov, 1986), a particle, resting with respect to the black hole at infinity, will move purely radially (in Boyer–Lindquist coordinates) and experience dragging in ϕ direction with Lense–Thirring velocity, while any additional impact to it in radial direction causes it to deflect in θ direction. Therefore, we already have the accretion pattern for cold matter. One should also expect that the slight heating of the gas far from the black hole, such that the gas molecules thermal velocity remains to be much smaller than the velocity of light, would not alter the flow picture drastically. Having had noticed all that, Shapiro came to calculation of the emerging bremsstrahlung radiation from almost cold infalling material. Further step was undertaken by Petrich, Shapiro & Teukolsky in 1988. They found analytical solution for arbitrary fast rotating Kerr black hole moving through the surrounding gas along its rotational axis, but for equation of state $p = \varrho$ with the sound velocity c_s equal to the light velocity everywhere, which looks very artificial.

Recently Beskin & Pidoprigora (1995) (further referred to as BP) considered the accretion for polytropic equation of state by slowly rotating black hole. They used small ratio of the rotational parameter a to the black hole mass \mathcal{M} as a small parameter to linearize hydrodynamic equations and to search for the solution as a small perturbation to the spherically symmetrical accretion by Schwarzschild black hole. The aim of the present work is to find the solution for the Kerr black hole with an *arbitrary value of a* . As in BP the black hole is supposed to be at rest with respect to the gas at infinity, flow is adiabatic and without angular momentum with respect to the black hole at infinity, matter has uniform temperature and density far from the black hole. Under these conditions flow must be axisymmetrical with the symmetry axis being the rotational axis of the black hole.

For solving the problem formulated we use the equation of flow lines equilibrium (stream equation), which is nonlinear differential equation of mixed type in flux function Φ and is of Grad–Shafranov type. Its general form for Kerr metric has been derived in Beskin & Pariev (1993). In section 2 basic quantities describing the flow are introduced and the equilibrium equation is formulated. As it is pointed above this equation must have solution corresponding to the accretion of the cold matter having zero pressure ($p = 0$). It occurred to be really true as have been verified in BP. The fact that usually in astrophysics substance far from the black hole has nonrelativistic temperature introduces small parameter to the problem — the ratio of the temperature at infinity to the rest mass of a particle, and allows us to search for the solution as a perturbation to the accretion of a cold matter. Stream equation can be linearized with respect to this parameter and the linearized equation occurred to be solvable analytically. This has been done in section 3. However, the solution for cold matter cannot be considered as a zero order solution for the expansion on the distances from the black hole comparable to the radius of the sonic surface (which is much greater than the radius of the event horizon), because the appearance of the sonic surface itself is essentially connected with the nonzero temperature of the matter. In section 4 we have constructed solution of the expanded stream equation in that region using as a zero order approximation the solution for Bondi accretion flow onto Newtonian gravitating centre of mass \mathcal{M} . Matching of the two solutions allows to determine the accretion pattern uniquely.

Construction of the analytical shock-free solution for the hydrodynamic accretion onto Kerr black hole leads us to the generalization for rapid rotation of the result of the work BP that neither the rotation of a black hole

nor its slow motion change drastically the nature of the accretion from the spherically symmetrical smooth flow.

2 Basic equations

Let us consider an axisymmetrical steady-state flow in the Kerr space-time with the metric in Boyer-Lindquist coordinates $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$:

$$ds^2 = -\alpha^2 dt^2 + g_{ik}(dx^i + \beta^i dt)(dx^k + \beta^k dt), \quad (1)$$

where

$$\alpha = \frac{\rho}{\Sigma} \sqrt{\Delta} \quad (2)$$

is the lapse function ($\alpha = \Delta = 0$ at the event horizon),

$$\beta^r = \beta^\theta = 0, \quad \beta^\phi = -\omega = -\frac{2a\mathcal{M}r}{\Sigma^2}, \quad (3)$$

ω is the angular velocity of Lense-Thirring precession, for all $i \neq k$ $g_{ik} = 0$,

$$g_{rr} = \rho^2/\Delta, \quad g_{\theta\theta} = \rho^2, \quad g_{\phi\phi} = \varpi^2, \quad (4)$$

and

$$\begin{aligned} \Delta &= r^2 + a^2 - 2\mathcal{M}r, & \rho^2 &= r^2 + a^2 \cos^2 \theta, \\ \Sigma^2 &= (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta, & \varpi &= \frac{\Sigma}{\rho} \sin \theta. \end{aligned} \quad (5)$$

Here \mathcal{M} is the mass of a black hole, $a = J/\mathcal{M}$ is the specific angular momentum of a black hole. Throughout all the paper we will adopt the notations used in Beskin & Pariev (1993) and in BP. We will use units where $c = 1$ and $G = 1$, that is all physical quantities have dimensions of some power of the unit of length. We follow the '3+1'-splitting of the space-time described in the book by Thorne et al. (1986). All physical quantities considered here are measured by Zero Angular Momentum Observers (ZAMO), rotating around the black hole at a constant radius r with Lense-Thirring angular velocity $d\phi/dt = \omega$. All 3-dimensional vectors and tensors are referred to 3-dimensional 'absolute' space with metric g_{ik} (4) orthogonal to the world lines of ZAMOs. It is convenient to use the following orthonormal triad in this 3-space

$$\mathbf{e}_{\hat{\phi}} = \frac{\sqrt{\Delta}}{\rho} \frac{\partial}{\partial r}, \quad \mathbf{e}_{\hat{\theta}} = \frac{1}{\rho} \frac{\partial}{\partial \theta}, \quad \mathbf{e}_{\hat{\phi}} = \frac{1}{\varpi} \frac{\partial}{\partial \phi}.$$

Axial symmetry and time independence allow us to introduce flux function $\Phi(r, \theta)$ defined as

$$\alpha n \mathbf{u}_P = \frac{1}{2\pi\varpi} (\nabla \Phi \times \mathbf{e}_{\hat{\phi}}). \quad (6)$$

Here n is the total particle density in the comoving reference frame, \mathbf{u}_P is the poloidal (that is in the directions of $\mathbf{e}_{\hat{\theta}}$ and $\mathbf{e}_{\hat{r}}$) component of the four velocity u^i . Due to the definition of $\Phi(r, \theta)$ by formula (6) the continuity equation $\nabla \cdot (\alpha n \mathbf{u}) = 0$ is satisfied automatically. Because of $(\mathbf{u} \nabla \Phi) = 0$ the flux function $\Phi(r, \theta)$ is constant along each stream line, and the quantity $\int_S d\Phi = \int_S \alpha n \mathbf{u} d\mathbf{S}$ is the total flux of the particles through an area S .

To avoid singularities on the rotational axis $\theta = 0$ flux function must satisfy two conditions $\Phi|_{\theta=0} = 0$ and $\frac{\partial \Phi}{\partial \theta} \Big|_{\theta=0} = \frac{\partial \Phi}{\partial \theta} \Big|_{\theta=\pi} = 0$. Note also that the value $-\Phi|_{\theta=\pi}$ is the total accretion rate (with positive sign).

Particularly, in the case of spherical symmetry $\Phi = \Phi_0(1 - \cos \theta)$, and the accretion rate is $-2\Phi_0$.

Following the tradition we adopt polytropic equation of state

$$p = k(s)n^\Gamma, \quad (7)$$

where p is the gas pressure, $1 < \Gamma \leq 5/3$ is the polytropic index, and $k(s)$ is the quantity depending only on the entropy per particle s . The flow is adiabatic, therefore, s is constant along stream lines $\Phi(r, \theta) = \text{const}$, so $s = s(\Phi)$. Then, entalpy per particle $\mu = (p + \varrho_m)/n$ (here ϱ_m is the specific internal energy of the gas) is

$$\mu = m + \frac{\Gamma}{\Gamma - 1} k(s) n^{\Gamma-1}, \quad (8)$$

where m is the mean rest mass of a particle. Accordingly, the sound velocity is

$$c_s^2 = \frac{1}{\mu} \left(\frac{\partial p}{\partial n} \right)_s = \frac{\Gamma k(s) n^{\Gamma-1}}{\mu}. \quad (9)$$

All thermodynamic quantities can be expressed via the functions in s and n only. Moreover, in our problem gas temperature and density are uniform on large distances from the black hole, so the entropy s is constant along all stream lines, that is constant everywhere $s = \text{const}$. Therefore, all thermodynamic quantities effectively depend on only one variable and can be expressed via each other given the value of s .

The equations of motion are provided by four components of the energy-momentum conservation law

$$T^{\hat{\mu}\hat{\nu}}_{;\hat{\nu}} = 0. \quad (10)$$

The 0-component of (10) leads to the energy conservation along each stream line (Bernoulli integral), which is manifested by that the following quantity E depends only on Φ

$$E(\Phi) = \mu(\alpha\gamma + \varpi\omega u_{\hat{\phi}}). \quad (11)$$

Here $\gamma^2 = 1 + u_P^2 + u_{\hat{\phi}}^2$ is the Lorentz factor of the flow. The $\hat{\phi}$ -component of (10) leads to the conservation of the z -component of the angular momentum along each stream line

$$L(\Phi) = \mu\varpi u_{\hat{\phi}}. \quad (12)$$

For the problem in hand $L(\Phi) = 0$, which means $u_{\hat{\phi}} = 0$. Hence, the gas corotates with ZAMOs with the angular velocity just equal to ω as seen by an infinitely distant observer. At infinity $\alpha = 1$, $\gamma = \gamma_{\infty} = 1$, and $\mu = \mu_{\infty} = \text{const}$, therefore, one can see from (11) that $E(\Phi) = E = \text{const}$ along all stream lines and, hence, everywhere. Now we can express all thermodynamic quantities and the velocity \mathbf{u} by means of $\Phi(r, \theta)$ and constants E and s only. Using definition (6) and obvious relation $\gamma^2 = 1 + u_P^2$ and bearing in mind that $L = 0$ one can obtain the following equation

$$\varpi^2 E^2 = \mu^2 \left(\Delta \sin^2 \theta + \frac{|\nabla \Phi|^2}{4\pi^2 n^2} \right), \quad (13)$$

which, being combined with the formula (8) for μ , provides an implicit expression for n in terms of $|\nabla \Phi|^2$, E , and s . Then, all other characteristics of the flow can be obtained from (6) and (8).

The remaining two space poloidal components of the equation (10) should be decomposed into parts along the stream line, i.e. perpendicular to $\nabla \Phi$, and perpendicular to the stream line, i.e. parallel to $\nabla \Phi$. Component along the stream line leads to the adiabatic condition $s = s(\Phi)$ and gives no new information, because even the form of the $T^{\mu\nu}$ for ideal fluid itself involves the conservation of the entropy (viscosity and thermoconductivity, if any, are taken into account by including additional terms in $T^{\mu\nu}$). Component perpendicular to the stream lines results in the equation of stream lines equilibrium — stream equation (or the Grad-Shafranov equation). With the terms containing second derivatives written out explicitly it looks as follows

$$\begin{aligned} D \left[\Delta \frac{\partial^2 \Phi}{\partial r^2} + \sin \theta \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial \Phi}{\partial \theta} \right) \right] + \frac{1}{\Delta(\partial \Phi / \partial r)^2 + (\partial \Phi / \partial \theta)^2} \left[\Delta^2 \left(\frac{\partial \Phi}{\partial r} \right)^2 \frac{\partial^2 \Phi}{\partial r^2} + \right. \\ \left. 2\Delta \frac{\partial \Phi}{\partial r} \frac{\partial \Phi}{\partial \theta} \frac{\partial^2 \Phi}{\partial r \partial \theta} + \left(\frac{\partial \Phi}{\partial \theta} \right)^2 \frac{\partial^2 \Phi}{\partial \theta^2} - \frac{1}{2} \frac{\partial \Delta}{\partial r} \frac{\partial \Phi}{\partial r} \left(\frac{\partial \Phi}{\partial \theta} \right)^2 \right] - \frac{1}{2} \frac{\partial}{\partial r} \left(\frac{\rho^2}{\Delta} \right) \frac{\Delta^2}{\rho^2} \frac{\partial \Phi}{\partial r} - \\ \frac{1}{2} \frac{1}{\rho^2} \frac{\partial \rho^2}{\partial \theta} \frac{\partial \Phi}{\partial \theta} - \frac{2\pi^2 n^2}{|\nabla \Phi|^2} \frac{\rho^2}{\mu^2} \nabla \Phi \cdot [E^2 \nabla(\varpi^2) - \mu^2 \nabla(\Delta \sin^2 \theta)] = 0, \end{aligned} \quad (14)$$

where

$$D = -1 + \frac{(\Gamma - 1)(\mu - m)}{m + (\mu - m)(2 - \Gamma)} \frac{4\pi^2 \Delta \sin^2 \theta n^2}{|\nabla \Phi|^2} = -1 + \frac{1}{u_P^2} \frac{c_s^2}{1 - c_s^2}. \quad (15)$$

Equation (14) with definition (15) and relations (8) and (13) contains actually only flux function $\Phi(r, \theta)$, two constants s and E as well as metric coefficients. It is elliptic in the subsonic region $D > 0$ and hyperbolic in the supersonic region $D < 0$. For the details of derivation of this equation see the work by Beskin & Pariev (1993). We need to find smooth solution of the equations (14), (15), (8) and (13) for the function $\Phi(r, \theta)$, considering boundary conditions at infinity.

3 Solution in the supersonic region

Let us first find the solution in the region far under the sonic surface $r = r_s$ for $r \ll r_s$, where we will treat it as being close to the solution for radially infalling dust. In the case of cold matter $k(s)$ is equal to 0, so $\mu = E = m$. By direct substitution into equation (14), expressing n in terms of Φ by means of the relation (13), it is possible to convince yourself that any arbitrary function $\Phi = \Phi(\theta)$ is the solution. This just means that the dust falling down to the black hole follows the trajectory $\theta = \text{const}$. At first, we need to specify this arbitrary function in order to determine zero order approximation for Φ . In our case of nonrelativistic temperature of the gas at infinity, sonic surface, at which the local sound velocity is equal to the velocity of the flow, is situated far from the black hole horizon $r_H = \mathcal{M} + \sqrt{\mathcal{M}^2 - a^2}$, in the region where relativistic effects are weak. Hence, the picture of the stream lines in the transsonic region only slightly deviates from that for the accretion onto Newtonian gravitating centre, and the main term in the expansion of Φ corresponds to spherically symmetrical flow, that is proportional to $1 - \cos\theta$. We will show in section 4 that the corrections to spherically symmetrical part of Φ in transsonic region are of order of the (\mathcal{M}/r_s) , while the corrections to angular dependent part are proportional to $(\mathcal{M}/r_s)^2$ and $(a/r_s)^2$.

We define small parameter of the expansion ϵ as the difference between the enthalpy per particle at infinity μ_∞ and the rest particle mass m

$$\mu_\infty = m(1 + \epsilon). \quad (16)$$

Far from the sound surface and the black hole $\mathbf{u} = 0$, hence, $E = \mu$. This gives us exact expression for the constant E

$$E = m(1 + \epsilon). \quad (17)$$

Introducing particle density at infinity n_∞ , the expression for enthalpy (8) is rewritten in the form

$$\mu - m = \epsilon m \left(\frac{n}{n_\infty} \right)^{\Gamma-1}. \quad (18)$$

It is well known that the accretion rate is determined by the condition of passing the solution for $n(r)$ smoothly through the sonic surface (Shapiro & Teukolsky, 1983). Consideration of spherically symmetrical Newtonian accretion problem (e.g. Shapiro & Teukolsky, 1983) results in the following accretion rate using our notations for enthalpy (16) and (18)

$$\Phi_0 = \Phi_{00} \frac{1}{\epsilon^{3/2}} (1 - \cos\theta), \quad (19)$$

where

$$\Phi_{00} = -2\pi n_\infty \mathcal{M}^2 \varphi(\Gamma) \quad (20)$$

and

$$\varphi(\Gamma) = \left(\frac{1}{2} \right)^{\frac{\Gamma+1}{2(\Gamma-1)}} \left(\frac{5-3\Gamma}{4} \right)^{-\frac{5-3\Gamma}{2(\Gamma-1)}} \frac{1}{(\Gamma-1)^{3/2}}. \quad (21)$$

This expression is valid for $1 < \Gamma < 5/3$. The case $\Gamma = 5/3$ needs special treatment and we will not concentrate on it though it is this case that is closest to real description of infalling matter if it is fully ionized. Thus, we have completely determined the function $\Phi_0(\theta)$, which was remained to be free in the case of perfectly cold matter. We accept the expression (19) as a zero order approximation to the solution. Note that, when $\epsilon \rightarrow 0$ and $n_\infty = \text{const}$, the accretion rate will become infinitely large. Further, by substitution the expression (19) into equation (13) zero order approximation for particle concentration n_0 can be found

$$n_0(r) = n_\infty \varphi^2(\Gamma) \epsilon^{-3/2} \frac{\mathcal{M}^2}{\sqrt{2\mathcal{M}r(r^2 + a^2)}}. \quad (22)$$

Also in zero order $\mu_0 = m$, $D_0 = -1$. Radius of the sonic surface is

$$r_{0s} = \mathcal{M} \frac{1}{\epsilon} \frac{5-3\Gamma}{4(\Gamma-1)}. \quad (23)$$

We see that $r_{0s} \gg \mathcal{M}$ for small ϵ indeed.

The main equation of the problem (14) may be expanded in parameter ϵ only when all values, characterizing the flow, deviate slightly from those for dust accretion. Therefore, matter needs having nonrelativistic temperature not only at infinity but also in the whole space. Let us check it. Substitution in expression (18) for n its zero order approximation (22) allows to find enthalpy with the first order accuracy

$$\mu - m = m (\varphi(\Gamma))^{\Gamma-1} \epsilon^{(5-3\Gamma)/2} \left(\frac{\mathcal{M}^3}{2r(r^2 + a^2)} \right)^{(\Gamma-1)/2}. \quad (24)$$

We see that if $\Gamma < 5/3$, for $\epsilon \rightarrow 0$ and $r \sim \mathcal{M}$ the correction to μ is small indeed and the expansion in ϵ is now becoming full substantiated. For $\Gamma = 5/3$ radius of the sonic surface $r_{0s} = \mathcal{M} \frac{3}{4} \sqrt{\frac{3}{2\epsilon}}$ is still much larger than \mathcal{M} , but the temperature in the vicinity of the event horizon is relativistic (for Schwarzschild black hole $T_H \approx 0.111 m$, Shapiro & Teukolsky (1983)) and the solution for dust accretion cannot be used as a zero order approximation.

Let us write for the flux function and particle density $\Phi = \Phi_0 + \Phi_1$ and $n = n_0 + n_1$, where $\Phi_1 \ll \Phi_0$ and $n_1 \ll n_0$, and perform the linearization of the equation (14) with definition (15) and relation (13), allowing to express n_1 via Φ_1 . Finally, after eliminating n_1 from linearized equation (14) and doing some algebra, stream equation can be reduced to the following simple equation in Φ_1

$$\begin{aligned} \frac{\partial^2 \Phi_1}{\partial r^2} + \frac{\partial \Phi_1}{\partial r} \frac{3r^2 + a^2}{2r(r^2 + a^2)} &= -\Phi_{00} \epsilon^{1-3\Gamma/2} \cos \theta \sin^2 \theta \times \\ \frac{a^2}{\mathcal{M}r(r^2 + a^2)} (\varphi(\Gamma))^{\Gamma-1} &\left(\frac{\mathcal{M}^3}{2r(r^2 + a^2)} \right)^{(\Gamma-1)/2}. \end{aligned} \quad (25)$$

The remarkable fact is the cancellation of the derivatives $\frac{\partial^2 \Phi_1}{\partial \theta^2}$, $\frac{\partial \Phi_1}{\partial \theta}$ and $\frac{\partial^2 \Phi_1}{\partial r \partial \theta}$, which makes equation (25) to be an ordinary differential equation and enables to find all its solutions. Note, that the equation (25) does not contain factor Δ , which vanishes at the event horizon r_H . This factor is cancelled out from all addenda of linearized equation (14), hence, the equation (25) has no singularities at the event horizon. There is no even the characteristic scale \mathcal{M} in the latter equation, though from physical point of view one can naturally expect it to be revealed. The only characteristic size is a , which can be much smaller than \mathcal{M} .

We are able to find easily general solution of equation (25), which can be written in the form such that the angular dependent part of Φ_1 tends to 0 for $r \rightarrow \infty$

$$\begin{aligned} \Phi_1 &= \Phi_{00} a^2 \cos \theta \sin^2 \theta 2^{-(\Gamma-1)/2} (\varphi(\Gamma))^{\Gamma-1} \mathcal{M}^{(3\Gamma-5)/2} \epsilon^{1-3\Gamma/2} \times \\ &\int_r^{+\infty} \frac{dr'}{\sqrt{r'(r'^2 + a^2)}} \int_{r_H}^{r'} \frac{dr''}{[r''(r''^2 + a^2)]^{\Gamma/2}} - A_1(\theta) \int_r^{+\infty} \frac{dr'}{\sqrt{r'(r'^2 + a^2)}} + A_2(\theta), \end{aligned} \quad (26)$$

where A_1 and A_2 are arbitrary functions in θ only. In order to match this solution with the solution in the transsonic region $r \sim r_s$ obtained in section 4 let us expand expression (26) in $a/r \ll 1$

$$\begin{aligned} \Phi_1 &= 2 \sqrt{\frac{\mathcal{M}}{r}} \left(1 - \frac{1}{10} \frac{a^2}{r^2} + \frac{1}{24} \frac{a^4}{r^4} - \dots \right) \left[\Phi_{00} C_0 \left(\frac{a}{\mathcal{M}} \right)^2 \cos \theta \sin^2 \theta (\varphi(\Gamma))^{\Gamma-1} \epsilon^{1-3\Gamma/2} 2^{3/2-2\Gamma} \right. \\ &\left. - \frac{A_1(\theta)}{\sqrt{\mathcal{M}}} \right] - \Phi_{00} \left(\frac{a}{\mathcal{M}} \right)^2 \cos \theta \sin^2 \theta (\varphi(\Gamma))^{\Gamma-1} 2^{-(\Gamma-1)/2} \epsilon^{1-3\Gamma/2} \frac{4}{(3\Gamma-2)(3\Gamma-1)} \times \\ &\left(\frac{\mathcal{M}}{r} \right)^{(3\Gamma-1)/2} \left(1 + \frac{3\Gamma-1}{3+3\Gamma} \frac{a^2}{r^2} - \frac{3}{8} \frac{3\Gamma-1}{7+3\Gamma} \frac{a^4}{r^4} - \dots \right) + A_2(\theta), \end{aligned} \quad (27)$$

where constant C_0 denotes the value of the integral

$$C_0 = \int_{r_H/(2\mathcal{M})}^{+\infty} \left[\xi \left(\xi^2 + \frac{a^2}{4\mathcal{M}^2} \right) \right]^{-\Gamma/2} d\xi.$$

It is also useful to find the ratio of the perturbation Φ_1 to zero order approximation $\Phi_{00} \epsilon^{-3/2}$. Retaining only leading terms in a^2/r^2 in expansion (27) one can obtain

$$\frac{\Phi_1}{\Phi_{00}} \epsilon^{3/2} \approx 2 \sqrt{\frac{\mathcal{M}}{r}} \left[C_0 \left(\frac{a}{\mathcal{M}} \right)^2 \cos \theta \sin^2 \theta (\varphi(\Gamma))^{\Gamma-1} 2^{3/2-2\Gamma} \epsilon^{(5-3\Gamma)/2} - \frac{A_1(\theta)}{\Phi_{00}} \epsilon^{3/2} \right] -$$

$$\left(\frac{a}{\mathcal{M}}\right)^2 \cos \theta \sin^2 \theta K(\Gamma) \left(\frac{\mathcal{M}}{r}\right)^{(3\Gamma-1)/2} \epsilon^{(5-3\Gamma)/2} + \frac{A_2(\theta)}{\Phi_{00}} \epsilon^{3/2}, \quad (28)$$

where we have denoted for brevity

$$K(\Gamma) = (\varphi(\Gamma))^{\Gamma-1} 2^{-(\Gamma-1)/2} \frac{4}{(3\Gamma-2)(3\Gamma-1)}. \quad (29)$$

We cannot now determine the functions $A_1(\theta)$ and $A_2(\theta)$. It will be done in section 4, where we obtain the solution of the stream equation (14) in the region $r \sim \mathcal{M}/\epsilon$ and match it to (28). At last, it is necessary to verify that $\Phi_1/\Phi_0 \ll 1$. Indeed, this is immediately seen from the equation (28) unless functions A_1 and A_2 have too high values, which is not the case as it will be shown in section 4.

4 Solution in the transsonic region

Now let us consider accretion flow far from the event horizon but still on the scales of order of the radius of the sound surface r_s . In this region zero approximation for particle density (22) can no longer be valid because when $r \sim \mathcal{M}/\epsilon$ main term in the difference $\varpi^2 E^2 - \mu^2 \Delta \sin^2 \theta$ standing for particle density in (13) $r^2 E^2 \sin^2 \theta - \mu^2 r^2 \sin^2 \theta$ vanishes for cold matter, when $E = \mu = m$, so to determine this difference correctly one must take into account thermal corrections to E and μ . Clearly, the consideration of the lowest order corrections will lead us to Bondi solution for the accretion in Newtonian gravitational potential of a point mass. We have already known the flux function for that solution Φ_0 given by formula (19). Therefore, zero order approximations for Φ in supersonic and transsonic regions are one and the same and are equal to Φ_0 . However, zero order approximations for entalpy and particle density are different.

To perform expansion of the equations (13), (18) and (14) with definition (15) in small parameter ϵ in transsonic region, first, we shall introduce some notations. Let us denote the ratio of the particle density to that at infinity as y

$$y(r, \theta) = \frac{n(r, \theta)}{n_\infty}.$$

Further it is necessary to introduce rescaled radius x . We define it as

$$r = \mathcal{M} \frac{x}{\epsilon} \quad (30)$$

Then, we will search for series in ϵ for y and Φ

$$y = y_0(x) + y_1(x, \theta) + y_2(x, \theta) + y_3(x, \theta) + \dots, \quad (31)$$

$$\Phi(r, \theta) = \Phi_{00} \epsilon^{-3/2} (1 - \cos \theta + f_1(x, \theta) + f_2(x, \theta) + f_3(x, \theta) + \dots), \quad (32)$$

where indexes $0, 1, 2, 3, \dots$ mark successive orders of expansion. For n and Φ dependent on θ sonic surface where $D = 0$ will no longer be spherical. Similar to the expansions of y and Φ we write an expansion of the radius of the sound surface $r_s(\theta)$

$$x_s(\theta) = x_{0s} + x_{1s}(\theta) + x_{2s}(\theta) + x_{3s}(\theta) + \dots, \quad (33)$$

where the zero order value corresponds to r_{0s} given by the expression (23)

$$x_{0s} = \frac{5 - 3\Gamma}{4(\Gamma - 1)}. \quad (34)$$

Because an asymptotical expansion of metric coefficients g_{ik} for $r \rightarrow \infty$ contains all successive orders in ϵ it is naturally to expect that y_1 and f_1 will be of order of ϵ , y_2 and f_2 of order of ϵ^2 , y_3 and f_3 of order of ϵ^3 , and so on. In further calculation dealing with series we shall follow this order of values that finally will have occurred to be true. The only characteristic size in the transsonic region is r_{0s} , therefore, θ - and x - derivatives of the smooth functions y_0, y_1, y_2, \dots , and f_1, f_2, \dots are of the same order and we shall regard in expansions $\frac{\partial f_1}{\partial x} \sim \frac{\partial f_1}{\partial \theta}$, $\frac{\partial y_2}{\partial x} \sim \frac{\partial y_2}{\partial \theta}$, and so on. In all subsequent expressions indexes $0, 1, 2, 3, \dots$ mark successive orders of expansion of the values considered, index s means that the quantity is taken at the nonperturbed sonic surface $x = x_{0s}$, particularly, $\mu = \mu_0 + \mu_1 + \mu_2 + \mu_3 + \dots$, $D = D_0 + D_1 + D_2 + \dots$, and similarly for other values.

Now zero order approximation for enthalpy is obtained by substituting $y_0(x)$ for n/n_∞ in the expression (18)

$$\mu_0 = m + \epsilon m y_0^{\Gamma-1}. \quad (35)$$

Expansion in ϵ of the relation (13) to the first order using (35), (17), and (20) results in the following expression

$$1 = y_0^{\Gamma-1} - \frac{1}{x} + \frac{\varphi^2(\Gamma)}{2y_0^2 x^4}, \quad (36)$$

which determines the function $y_0(x)$. Differentiating (36) one can obtain

$$\frac{dy_0}{dx} = \frac{y_0}{x} \frac{2\varphi^2(\Gamma) - x^3 y_0^2}{(\Gamma-1)y_0^{\Gamma+1} x^4 - \varphi^2(\Gamma)}. \quad (37)$$

At the point $x = x_{0s}$, where the denominator in r.h.s. of (37) vanishes, the solution $y_0(x)$ must be regular. This means that at $x = x_{0s}$ the numerator must vanish too. Hence, we have three relations among values x_{0s} , $y_{0s} = y_0(x_{0s})$, and $\varphi(\Gamma)$; they are equation (36), the equality to zero of the denominator in (37)

$$\varphi^2(\Gamma) = (\Gamma-1)y_{0s}^{\Gamma+1} x_{0s}^4, \quad (38)$$

and the equality to zero of the numerator in (37)

$$\varphi^2(\Gamma) = \frac{1}{2} x_{0s}^3 y_{0s}^2. \quad (39)$$

Solution of these three relations gives the value of $\varphi(\Gamma)$ just equal to that given by formula (21) used in section 3, the value of x_{0s} equal to (34), and the following value for y_{0s}

$$y_{0s} = \left(\frac{2}{5-3\Gamma} \right)^{1/(\Gamma-1)}. \quad (40)$$

With the relations (38), (39) and (40) the equation (36) determines two functions $y_0(x)$ passing smoothly through the sonic point $x = x_{0s}$, one corresponding to Newtonian accretion onto gravitating point object, the other to Newtonian ejection from gravitating object (Shapiro & Teukolsky, 1983). Asymptotic of the accretion solution for $x \rightarrow \infty$ is $y_0 \rightarrow 1 + \frac{1}{\Gamma-1} \frac{1}{x}$, of the ejection solution $y_0 \sim x^{-2}$; asymptotics for $x \rightarrow 0$ are: for the accretion solution $y_0 \rightarrow \frac{\varphi(\Gamma)}{\sqrt{2}} x^{-3/2}$, for the ejection solution $y_0 \sim x^{-1/(\Gamma-1)}$. We plotted functions $y_0(x)$ on Fig. 1 for $\Gamma = 4/3$. As it can be seen from the asymptotical behaviour of these two functions their relative

location for all values $1 < \Gamma < 5/3$ is the same as on Fig. 1. In order to determine the derivative $\left. \frac{dy_0}{dx} \right|_{x=x_{0s}}$ we

expand the relation (36) up to the second order in the vicinity of the point $x = x_{0s}$, $y = y_{0s}$. Coefficients in the first order expansion identically vanish due to the conditions (38) and (39). As a result, we obtain quadratic equation in the derivative sought, which has two solutions, one corresponding to the accretion curve $y_0(x)$, the other to the ejection curve of Bondi solution. Note that no additional conditions should be imposed in order to ensure the existence and continuity of all higher derivatives of $y_0(x)$ at the sonic point. Because r.h.s. of the equation (36) is function analytic in both x and y_0 in the neighbourhood of $x = x_{0s}$ two functions $y_0(x)$ determined implicitly by the equation (36) are analytic too. Since the lines $y_0 = y_0(x)$ are lines of constant value of the r.h.s. of the equation (36), they cannot intersect each other save the sonic point. Therefore, deducing from the asymptotical behaviour of the accretion and ejection solutions, one must choose for the derivative of $y_0(x)$ at the sonic point larger value from two. Thus, we obtain

$$\left. \frac{dy_0}{dx} \right|_{x=x_{0s}} = 2 \frac{\Gamma-1}{\Gamma+1} \left(\frac{2}{5-3\Gamma} \right)^{\Gamma/(\Gamma-1)} \left(-4 + \sqrt{10-6\Gamma} \right). \quad (41)$$

Zero order of D given by the formula (15) is

$$D_0 = -1 + \frac{\Gamma-1}{\varphi^2(\Gamma)} y_0^{\Gamma+1} x^4. \quad (42)$$

At the sonic point $D_0(x_{0s}) = 0$. In the stream equation (14)

$$L_\theta \Phi_0 = \sin \theta \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial \Phi_0}{\partial \theta} \right) = 0$$

and substitution of zero order values in all other terms of equation (14) leads after cancelling out the factor $\cos \theta$ to the relation (36). Hence, in the zero order approximation stream equation (14) is satisfied automatically.

4.1 First order corrections

Now let us turn to the first order corrections. From (18) we obtain

$$\mu_1 = \epsilon m y_0^{\Gamma-1} \frac{y_1}{y_0} (\Gamma - 1). \quad (43)$$

Expansion of the quantity $\frac{4\pi^2 n^2}{|\nabla\Phi|^2}$ entering both the relation (13) and the equation (14) up to the first order is

$$\left. \frac{4\pi^2 n^2}{|\nabla\Phi|^2} \right|_0 + \left. \frac{4\pi^2 n^2}{|\nabla\Phi|^2} \right|_1 = \frac{\epsilon x^2 y_0^2}{\mathcal{M}^2 \varphi^2(\Gamma) \sin^2 \theta} \left(1 + 2 \frac{y_1}{y_0} - \frac{2}{\sin \theta} \frac{\partial f_1}{\partial \theta} \right). \quad (44)$$

Because of the vanishing of the leading term $\sim 1/\epsilon^2$ in the relation (13) we need to expand it to the second order and, hence, one might expect that the rotational parameter a will be involved in the first order approximation, because it enters quadratically into the expressions (5) for ϖ^2 and Δ . However, calculation shows that the terms containing a^2 are cancelled out from the both sides of the equation (13), thus resulting in the following expression for y_1 through f_1

$$2 \frac{y_1}{y_0} \left((\Gamma - 1) y_0^{\Gamma-1} - \frac{\varphi^2(\Gamma)}{x^4 y_0^2} \right) = \epsilon (1 + 3 y_0^{2\Gamma-2} - 4 y_0^{\Gamma-1}) - \frac{\varphi^2(\Gamma)}{x^4 y_0^2} \frac{2}{\sin \theta} \frac{\partial f_1}{\partial \theta}. \quad (45)$$

Further, we have to linearize equation (14) using (43) and (44) and insert into it the expression (45) for y_1 . The θ -component of the expression embraced by the square brackets in the last term of l.h.s. of (14) $\frac{E^2}{\mu^2} \frac{\partial \varpi^2}{\partial \theta} - 2\Delta \sin \theta \cos \theta$ must be expanded to the second order (because of the vanishing leading term in this difference) and, again, a^2 is cancelled, and the linearization of the stream equation does not contain a^2 . For D it is enough to use its zero order D_0 ; in the linearized last term in l.h.s. of (14) y_1 stands only multiplied by the factor $\frac{2}{y_0} \left((\Gamma - 1) y_0^{\Gamma-1} - \frac{\varphi^2(\Gamma)}{x^4 y_0^2} \right)$ that allows it to be directly replaced by the r.h.s. of expression (45). This replacement results in the cancellation of the free term in stream equation. Finally, dividing (14) by $\Phi_{00} \epsilon^{-3/2}$ we obtain the following linear second order differential equation in f_1

$$D_0 x^2 \frac{\partial^2 f_1}{\partial x^2} + (D_0 + 1) \sin \theta \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial f_1}{\partial \theta} \right) - \left(2x - \frac{x^4 y_0^2}{\varphi^2(\Gamma)} \right) \frac{\partial f_1}{\partial x} = 0, \quad (46)$$

where D_0 is given by the expression (42).

Equation (46) is an equation with separable variables and angular operator

$$L_\theta = \sin \theta \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right) \quad (47)$$

is the same as in the linearizations of the stream equation considered in BP and by Petrich, Shapiro & Teukolsky (1988). Particle flux density must be finite at the rotational axis. As it is seen from the expression (6) for $n\mathbf{u}_P$ this leads to that the quantity $\frac{1}{\sin \theta} \frac{\partial f_1}{\partial \theta}$ must be finite at $\theta = 0$ and $\theta = \pi$. Besides, it is obvious that at $\theta = 0$ total flux must be equal to 0, so one more boundary condition for f_1 is $f_1|_{\theta=0} = 0$. It can be proved that the eigenvalue problem for operator L_θ with these boundary conditions has eigenvalues $C_n = -n(n+1)$, where $n = 0, 1, 2, 3, \dots$, with corresponding eigenfunctions

$$Q_n = \frac{2^n n! (n-1)!}{(2n)!} \sin^2 \theta P'_n(\cos \theta),$$

where prime denotes differentiation of the Legendre polynomial $P_n(\xi)$ with respect to ξ and normalizing factor is chosen the same as in BP to make the comparison of our results to the results by BP more direct. First four eigenfunctions are

$$Q_0 = 1 - \cos \theta, \quad Q_1 = \sin^2 \theta, \quad Q_2 = \sin^2 \theta \cos \theta, \quad Q_3 = \sin^2 \theta \left(\cos^2 \theta - \frac{1}{5} \right). \quad (48)$$

We look for the solution of the equation (46) in the form

$$f_1 = \sum_{n=0}^{\infty} \epsilon g_{1n}(x) Q_n(\theta). \quad (49)$$

Then $g_{1n}(x)$ must satisfy the following equation

$$D_0 x^2 \frac{d^2 g_{1n}}{dx^2} + \left(\frac{x^4 y_0^2}{\varphi^2(\Gamma)} - 2x \right) \frac{dg_{1n}}{dx} - (D_0 + 1)n(n+1)g_{1n} = 0 \quad (50)$$

with boundary conditions that $g_{1n}(x)$ must be regular at the sonic point where $D_0 = 0$ and be finite at $x \rightarrow \infty$. Two linearly independent solutions of (50) have asymptotics $\sim x^{n+1}$ and $\sim x^{-n}$, so we must choose the latter solution. Because of the relation (39) it follows from equation (50) that $n(n+1)g_{1n}(x_{0s}) = 0$. For $n \neq 0$ it means that $g_{1n}(x) \equiv 0$, while for $n = 0$ one obvious solution of uniform equation (50) is $g_{10} = \text{const}$. Second linearly independent solution has asymptotic $g_{10} \sim x$ for $x \rightarrow \infty$ and must be ruled out. Thus, first order correction to the flux function f_1 has the form

$$f_1 = \epsilon g_{10}(1 - \cos \theta). \quad (51)$$

To complete construction of the first order approximation it remains to calculate y_1 according to the equation (45). We see that y_1 does not depend on θ . Correction to particle density $y_1(x)$ must be smooth function in the vicinity of the sonic point $x = x_{0s}$, where according to (38) l.h.s. of the equation (45) vanishes. Therefore, r.h.s. must vanish too, which allows us to determine the value of g_{10}

$$g_{10} = \frac{3}{4} \frac{3\Gamma + 1}{5 - 3\Gamma}. \quad (52)$$

Now y_1 is given by the formula

$$y_1 = \epsilon y_0 \frac{1 + 3y_0^{2\Gamma-2} - 4y_0^{\Gamma-1} - 2g_{10} \frac{\varphi^2(\Gamma)}{x^4 y_0^2}}{2(\Gamma-1)y_0^{\Gamma-1} - 2 \frac{\varphi^2(\Gamma)}{x^4 y_0^2}}. \quad (53)$$

Both numerator and denominator in (53) are analytic functions in y_0 and x , therefore, $y_1(x)$ is also analytic function in the neighbourhood of the sonic point, and no additional constraints should be imposed in order of continuity of the derivatives of y_1 . The value of y_1 at the sonic point is

$$y_1|_{x=x_{0s}} = \frac{\epsilon y_{0s}}{2(5-3\Gamma)\sqrt{10-6\Gamma}} \left[\frac{21\Gamma-5}{\Gamma+1} (-4 + \sqrt{10-6\Gamma}) + 6(3\Gamma+1) \right]. \quad (54)$$

Asymptotical behaviour for $x \rightarrow \infty$ is $y_1 \rightarrow \frac{\epsilon}{x(\Gamma-1)}$. Function $y_1(x)$ is plotted on Fig. 2 for some values of Γ .

What we actually have obtained considering first order approximation are the corrections to Newtonian accretion rate and particle density in the problem of spherically symmetrical accretion onto a Schwarzschild black hole. As long as a is not contained in the results they coincide with those for a Schwarzschild black hole. Indeed, exact analytical solution for the accretion onto a Schwarzschild black hole (e.g., Shapiro & Teukolsky, 1983) in our notations can be written as

$$\Phi = \Phi_{sw}(1 - \cos \theta)$$

with

$$\Phi_{sw} = -2\pi n_{\infty} \mathcal{M}^2 \epsilon^{-1/(\Gamma-1)} \frac{1}{4} \left(\frac{\mathcal{M}}{2r_s} \right)^{\frac{5-3\Gamma}{2(\Gamma-1)}} \frac{1}{\left(\Gamma - 1 + \frac{\mathcal{M}}{2r_s} (2 - 3\Gamma) \right)^{1/(\Gamma-1)}}, \quad (55)$$

where the value $\frac{\mathcal{M}}{2r_s} \sim \epsilon$ should be obtained by solving cubic equation

$$(1 + \epsilon)^2 \left(1 - \frac{3\Gamma-2}{\Gamma-1} \frac{\mathcal{M}}{2r_s} \right)^2 = \left(1 - 3 \frac{\mathcal{M}}{2r_s} \right)^3. \quad (56)$$

For the rescaled radius of the sonic surface $x_{sw} = \epsilon r_s / \mathcal{M}$ expansion of the solution of equation (56) in terms of ϵ up to ϵ^3 is

$$x_{sw} = \frac{5-3\Gamma}{4(\Gamma-1)} \left[1 + \epsilon \frac{30\Gamma-9\Gamma^2-13}{2(5-3\Gamma)^2} + \epsilon^2 \frac{27}{4} \frac{(3\Gamma^2-14\Gamma+3)(\Gamma-1)^2}{(5-3\Gamma)^4} - \epsilon^3 \frac{27(\Gamma-1)^2}{8(5-3\Gamma)^6} (3\Gamma+1)(9\Gamma^3-75\Gamma^2+103\Gamma-53) \right]. \quad (57)$$

For all $1 < \Gamma < 5/3$ the successive terms in the series (57) have alternating signs: $x_{sw1} > 0$, $x_{sw2} < 0$, $x_{sw3} > 0$. Substitution of the expression (57) into formula (55) results in series for the accretion rate

$$\Phi_{sw} = \Phi_{00} \epsilon^{-3/2} \left(1 + \epsilon \frac{3}{4} \frac{3\Gamma+1}{5-3\Gamma} - \epsilon^2 \frac{3}{32} \frac{135\Gamma^3-99\Gamma^2-207\Gamma+139}{(5-3\Gamma)^3} + \epsilon^3 \frac{8505\Gamma^5-9153\Gamma^4-37530\Gamma^3+89226\Gamma^2-71583\Gamma+20279}{128(5-3\Gamma)^5} \right). \quad (58)$$

First order term in (58) coincides with the expression for g_{10} (52) and is positive, which means enhanced accretion rate. Term proportional to ϵ^2 is positive for $1 < \Gamma < 1.3037$ and negative for $1.3037 < \Gamma < 5/3$, term proportional to ϵ^3 is positive for $1.1558 < \Gamma < 5/3$ and negative for $1 < \Gamma < 1.1558$.

At last, let us show that the correction to the radius of the sonic surface x_{1s} (see expansion (33)) does not depend on θ and is equal to x_{sw1} from (57). In general sonic surface is located where $D = 0$. First order of D given by the formula (15) is

$$D_1 = (\Gamma-1) \frac{y_0^{\Gamma+1} x^4}{\varphi^2(\Gamma)} \left((\Gamma+1) \frac{y_1}{y_0} - (2-\Gamma) \epsilon y_0^{\Gamma-1} - \frac{2\epsilon}{x} - 2\epsilon g_{10} \right) \quad (59)$$

and does not depend on θ . Correction x_{1s} is determined from the equation

$$\left. \frac{dD_0}{dx} \right|_{x=x_{0s}} x_{1s} + D_1(x_{0s}) = 0.$$

Its solution is readily obtained using expressions (41) and (54)

$$x_{1s} = \epsilon \frac{30\Gamma-9\Gamma^2-13}{8(\Gamma-1)(5-3\Gamma)} \quad (60)$$

and coincides with x_{sw1} .

4.2 Second order corrections

In order to match the solution in the supersonic region we need to consider further orders of expansion in the transonic region. Finding y_2 and f_2 we will act similarly as when we deal with y_1 and f_1 . Let us denote by $\hat{a} = a/\mathcal{M}$ dimensionless value of a . From (18) we obtain

$$\mu_2 = \epsilon m y_0^{\Gamma-1} (\Gamma-1) \left(\frac{y_2}{y_0} + \frac{\Gamma-2}{2} \frac{y_1^2}{y_0^2} \right),$$

further,

$$\left. \frac{4\pi^2 n^2}{|\nabla\Phi|^2} \right|_2 = \frac{\epsilon x^2}{\mathcal{M}^2 \varphi^2(\Gamma) \sin^2 \theta} \left(2y_0 y_2 + y_1^2 + \epsilon^2 \frac{\hat{a}^2}{x^2} \cos^2 \theta y_0^2 - 4\epsilon g_{10} y_0 y_1 - \frac{2}{\sin \theta} \frac{\partial f_2}{\partial \theta} y_0^2 + 3\epsilon^2 g_{10}^2 y_0^2 \right).$$

Third order of expansion of relation (13) gives an expression for y_2 as a function of unknown f_2

$$\begin{aligned} 2 \frac{y_2}{y_0} \left(y_0^{\Gamma-1} (\Gamma-1) - \frac{\varphi^2(\Gamma)}{x^4 y_0^2} \right) &= \epsilon^2 \left[\frac{\hat{a}^2}{x^2} \frac{\varphi^2(\Gamma)}{y_0^2 x^4} + 2 \frac{\hat{a}^2}{x^2} \cos^2 \theta (1 - y_0^{\Gamma-1}) + 2 y_0^{2\Gamma-2} (y_0^{\Gamma-1} - 1) \right. \\ &\quad \left. - g_{10} (g_{10} + 4 y_0^{\Gamma-1}) \frac{\varphi^2(\Gamma)}{y_0^2 x^4} \right] + 2\epsilon \frac{y_1}{y_0} \left[y_0^{2\Gamma-2} (\Gamma-1) - 2(\Gamma-1) y_0^{\Gamma-1} + 2 y_0^{\Gamma-1} \frac{\varphi^2(\Gamma)}{y_0^2 x^4} \right. \\ &\quad \left. + 2 g_{10} \frac{\varphi^2(\Gamma)}{y_0^2 x^4} \right] - \frac{y_1^2}{y_0^2} \left[(\Gamma-1)(\Gamma-2) y_0^{\Gamma-1} + 3 \frac{\varphi^2(\Gamma)}{y_0^2 x^4} \right] - \frac{2}{\sin \theta} \frac{\partial f_2}{\partial \theta} \frac{\varphi^2(\Gamma)}{y_0^2 x^4}. \end{aligned} \quad (61)$$

Note that when obtaining second order of the stream equation (14) we need to retain only leading term D_0 in the series for D because both Φ_0 and Φ_1 do not depend on r and $L_\theta \Phi_0 = L_\theta \Phi_1 = 0$. Therefore, partial differential equation for f_2 changes its type at the surface $x = x_{0s}$ as well as equation (46) for f_1 does. Again, y_2 appears in the second order of the stream equation only multiplied by $\frac{2}{y_0} \left(y_0^{\Gamma-1} (\Gamma-1) - \frac{\varphi^2(\Gamma)}{x^4 y_0^2} \right)$, which allows it to be excluded by means of expression (61). Finally, dividing (14) by $\Phi_{00} \epsilon^{-3/2}$ one can obtain the following linear second order partial differential equation in f_2

$$D_0 x^2 \frac{\partial^2 f_2}{\partial x^2} + (D_0 + 1) \sin \theta \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial f_2}{\partial \theta} \right) - \left(2x - \frac{y_0^2 x^4}{\varphi^2(\Gamma)} \right) \frac{\partial f_2}{\partial x} + \epsilon^2 \frac{\hat{a}^2}{x^2} \cos \theta \sin^2 \theta \left(1 - \frac{2}{x} \frac{y_0^2 x^4}{\varphi^2(\Gamma)} \right) = 0. \quad (62)$$

Variables in equation (62) can be separated and angular operator coincides with that in equation (46). Moreover, equation (62) differs from equation (46) only by that the former contains free term, while the latter does not. Equation (62) coincides with the limit $c_s^2 \ll 1$ of the equation (52) in BP, which is the linearization of the stream equation in small parameter a/\mathcal{M} , so the results of BP concerning the case $c_s^2 \ll 1$ can be applied to the equation (62). Similar to the equation (46) we seek for the solution in the form

$$f_2 = \sum_{n=0}^{\infty} \epsilon^2 g_{2n}(x) Q_n(\theta).$$

Then, radial functions $g_{2n}(x)$ for all $n \neq 2$ must satisfy exactly the same equation as (50) with identical boundary conditions. This immediately allows to conclude that $g_{2n} \equiv 0$ for all $n \neq 0, 2$ and $g_{20} = \text{const}$. As for g_{22} it must satisfy the following equation

$$-D_0 \frac{d^2 g_{22}}{dx^2} + \left(\frac{2}{x} - \frac{1}{x^2} \frac{y_0^2 x^4}{\varphi^2(\Gamma)} \right) \frac{dg_{22}}{dx} + \frac{6}{x^2} (D_0 + 1) g_{22} = \frac{\hat{a}^2}{x^4} \left(1 - \frac{2}{x} \frac{y_0^2 x^4}{\varphi^2(\Gamma)} \right), \quad (63)$$

which is identical to equation (57) in BP. The requirement of regularity of $g_{22}(x)$ at nonperturbed sound surface $D_0 = 0$ leads to the condition

$$g_{22}(x_{0s}) = -\frac{\hat{a}^2}{2x_{0s}^2}. \quad (64)$$

Another boundary condition for g_{22} is $g_{22} \sim x^{-2}$ when $x \rightarrow \infty$. Similarly to BP we introduce function $G(x)$ such that the solution of (63) satisfying boundary conditions is written in the form

$$g_{22}(x) = -G(x) \frac{\hat{a}^2}{x_{0s}^2} \left(\frac{x}{x_{0s}} \right)^{(1-3\Gamma)/2},$$

where $G(x_{0s}) = 1/2$ and $G(x) \sim x^{(3\Gamma-5)/2}$ for $x \rightarrow \infty$. The dependencies $G(x)$ found numerically by integrating (63) with boundary conditions are shown on Fig. 3, which is actually the same as Fig. 1 in BP. Thus, second order correction to the flux function f_2 has the form

$$f_2 = \epsilon^2 g_{20} (1 - \cos \theta) - \epsilon^2 G(x) \frac{\hat{a}^2}{x_{0s}^2} \left(\frac{x}{x_{0s}} \right)^{(1-3\Gamma)/2} \sin^2 \theta \cos \theta. \quad (65)$$

Similar to constant g_{10} constant g_{20} should be determined from the requirement of smoothness of the value y_1 at the sonic surface $x = x_{0s}$. Inserting into expression (61) value for f_2 from (65) we see that

$$\frac{1}{\sin \theta} \frac{\partial f_2}{\partial \theta} = \epsilon^2 g_{20} + \epsilon^2 g_{22}(x) (3 \cos^2 \theta - 1).$$

Therefore, expression (61) contains terms proportional to $\cos^2 \theta$ and terms dependent only on x . For evaluating r.h.s. of (61) at the nonperturbed sonic surface we use expressions (64) for $g_{22}(x_{0s})$, (54) for $y_1(x_{0s})$, (40) for $y_0(x_{0s})$, and (52) for g_{10} . As a result, terms proportional to $\cos^2 \theta$ vanish, while the requirement for the rest addenda to be equal to 0 determines the value of g_{20}

$$g_{20} = -\frac{3}{32} \frac{135\Gamma^3 - 99\Gamma^2 - 207\Gamma + 139}{(5 - 3\Gamma)^3}, \quad (66)$$

which actually coincides with the second order term in the expansion of Φ_{sw} (58). This result is obvious because in the case $a^2 = 0$ when $g_{22} = 0$ and $f_2 = \epsilon^2 g_{20}(1 - \cos \theta)$ we deal with the accretion onto a Schwarzschild black hole. Now the construction of the second order approximation is completed.

Now we are able to match the solution in the supersonic region obtained in section 3 with the solution in the transonic region. For this purpose we should consider the behaviour of $g_{22}(x)$ for $x \rightarrow 0$. Expanding the relation (36) in the vicinity of $x = 0$ one can find two leading terms in the asymptotical series for $y_0(x)$

$$y_0 = \frac{\varphi(\Gamma)}{\sqrt{2}} x^{-3/2} \left[1 + \frac{(\varphi(\Gamma))^{\Gamma-1}}{2^{(\Gamma+1)/2}} x^{(5-3\Gamma)/2} + o\left(x^{(5-3\Gamma)/2}\right) \right].$$

Substitution of this expression into r.h.s. of (63) allows us to find first nonzero term in the expansion of r.h.s. of (63). Hence, in the limit $x \rightarrow 0$ equation (63) takes the form

$$\frac{d^2 g_{22}}{dx^2} + \frac{3}{2x} \frac{dg_{22}}{dx} = -\hat{a}^2 \frac{(\varphi(\Gamma))^{\Gamma-1}}{2^{(\Gamma-1)/2}} x^{-3(\Gamma+1)/2}.$$

General solution of the last equation is

$$g_{22} = C_1 x^{-1/2} + C_2 - \hat{a}^2 x^{(1-3\Gamma)/2} K(\Gamma), \quad (67)$$

where the constant $K(\Gamma)$ occurs to be just equal to that given by (29) and constants C_1 and C_2 are uniquely determined by boundary conditions for g_{22} . Since $g_{22}(x)$ satisfies boundary condition (64) at the sonic point, constants C_1 and C_2 are of order of \hat{a}^2 and it is convenient to write them explicitly $C_1 = c_1(\Gamma)\hat{a}^2$, $C_2 = c_2(\Gamma)\hat{a}^2$. Therefore, in the inner region $x \sim \epsilon$ the third term in the expression (67) is dominant. This conclusion is confirmed by numerical computation of $g_{22}(x)$, the results of which are plotted on Fig. 3. At $x \rightarrow 0$ function $G(x)$ tends to a constant value, which demonstrates that the first two terms in the expression (67) become negligible compared to the third one. Limiting value of $G(x)$ is derived from the last term of (67)

$$\lim_{x \rightarrow 0} G(x) = \frac{4}{(3\Gamma - 2)(3\Gamma - 1)(\Gamma - 1)2^\Gamma}$$

and is in good agreement with the numerical results. Limit $r \gg \mathcal{M}$ of the inner solution (28) when expressed in terms of rescaled variable x takes the form

$$\begin{aligned} \frac{\Phi_1}{\Phi_{00}} \epsilon^{3/2} \Big|_{\text{in}} &= 2C_0 \hat{a}^2 \cos \theta \sin^2 \theta (\varphi(\Gamma))^{\Gamma-1} 2^{3/2-2\Gamma} x^{-1/2} \epsilon^{3(2-\Gamma)/2} - 2 \frac{A_1(\theta)}{\Phi_{00} \sqrt{\mathcal{M}}} x^{-1/2} \epsilon^2 - \\ &\hat{a}^2 \cos \theta \sin^2 \theta K(\Gamma) x^{(1-3\Gamma)/2} \epsilon^2 + \frac{A_2(\theta)}{\Phi_{00}} \epsilon^{3/2}. \end{aligned} \quad (68)$$

For outer solution (32) combining (51), (65) we have

$$\begin{aligned} \frac{\Phi_1}{\Phi_{00}} \epsilon^{3/2} \Big|_{\text{out}} &= \epsilon^2 C_1 \cos \theta \sin^2 \theta x^{-1/2} + \epsilon^2 C_2 \cos \theta \sin^2 \theta - \epsilon^2 \hat{a}^2 x^{(1-3\Gamma)/2} K(\Gamma) \cos \theta \sin^2 \theta \\ &+ (1 - \cos \theta)(\epsilon g_{10} + \epsilon^2 g_{20}) + O(\epsilon^3), \end{aligned} \quad (69)$$

where symbol $O(\epsilon^\alpha)$ denotes terms proportional to ϵ^α . In order to match expression (68) to (69) we must choose the following values for functions $A_1(\theta)$ and $A_2(\theta)$

$$\begin{aligned} A_1(\theta) &= \Phi_{00} \sqrt{\mathcal{M}} \hat{a}^2 \cos \theta \sin^2 \theta \left[C_0 (\varphi(\Gamma))^{\Gamma-1} 2^{3/2-2\Gamma} \epsilon^{1-3\Gamma/2} - \frac{1}{2} c_1(\Gamma) \right] + O(\epsilon), \\ A_2(\theta) &= \Phi_{00} \hat{a}^2 \epsilon^{1/2} c_2(\Gamma) \cos \theta \sin^2 \theta + \Phi_{00} (1 - \cos \theta) \epsilon^{-1/2} (g_{10} + \epsilon g_{20}) + O(\epsilon^{3/2}). \end{aligned}$$

With these expressions solution (26) in the supersonic region becomes fully determined and can be expressed by the following formula

$$\begin{aligned} \frac{\Phi_1}{\Phi_{00}} \epsilon^{3/2} &= -\frac{a^2}{\mathcal{M}^2} \cos \theta \sin^2 \theta (\varphi(\Gamma))^{\Gamma-1} 2^{1-2\Gamma} \epsilon^{(5-3\Gamma)/2} \int_{r/(2\mathcal{M})}^{+\infty} \frac{d\xi'}{\sqrt{\xi'(\xi'^2 + \hat{a}^2/4)}} \times \\ &\int_{\xi'}^{+\infty} \frac{d\xi''}{[\xi''(\xi''^2 + \hat{a}^2/4)]^{\Gamma/2}} + \frac{a^2}{\mathcal{M}^2} \cos \theta \sin^2 \theta \frac{c_1(\Gamma)}{2^{3/2}} \epsilon^{3/2} \int_{r/(2\mathcal{M})}^{+\infty} \frac{d\xi'}{\sqrt{\xi'(\xi'^2 + \hat{a}^2/4)}} + \\ &\epsilon^2 \frac{a^2}{\mathcal{M}^2} c_2(\Gamma) \cos \theta \sin^2 \theta + (1 - \cos \theta)(\epsilon g_{10} + \epsilon^2 g_{20}) + O(\epsilon^{5/2}). \end{aligned} \quad (70)$$

Firstly, we see that $\Phi_1 \ll \Phi_0$. Secondly, dominating term in (70) is the first one and, since we deal only with the first order approximation to Φ , we cannot be sure that higher order terms are negligible in comparison with the rest addenda in (70). At the same time, only terms dependent on θ like $Q_0 = 1 - \cos \theta$ contribute to the total accretion rate, and because the total accretion rate must be some constant value independent of the radius these terms do not depend on r in all orders in ϵ . Hence, these terms are the same in the supersonic region and in the transsonic region and are determined correctly in the formula (70) up to the $O(\epsilon^3)$. Finally, we write down explicitly the solution in the transsonic region which matches the solution (70)

$$\Phi = \Phi_{00}\epsilon^{-3/2}[1 + (\epsilon g_{10} + \epsilon^2 g_{20})(1 - \cos \theta) + \epsilon^2 g_{22}(r) \cos \theta \sin^2 \theta + O(\epsilon^3)]. \quad (71)$$

On Fig. 4 we plotted the picture of the stream lines corresponding to the flux function given by formula (70) for $r \ll r_{0s}$ and by formula (71) for $r \gg r_{0s}$. Stream lines concentrate toward the equatorial plane of a black hole.

To find second order correction $x_{2s}(\theta)$ to the shape of the sonic surface one should expand the condition $D(x_s) = 0$ up to the second order in ϵ similarly to that was done when determining first order correction x_{1s} (60). However, because the second order solution in the transsonic region is actually the same as the solution for $a \ll \mathcal{M}$ and $r_s \gg \mathcal{M}$ obtained in BP, it is much easier to notice that the shape of the sonic surface up to the second order in ϵ coincides with that found in BP for the case $a \ll \mathcal{M}$ and $\mathcal{M}/r_s \ll 1$. So, rewriting formula (74) from BP and adding term proportional to ϵ^2 from expression for x_{sw} (57), we obtain in our notations

$$x_{2s} = \epsilon^2 \frac{a^2}{\mathcal{M}^2} \frac{2(\Gamma - 1)}{(5 - 3\Gamma)^2} [(6\Gamma - 10) \cos^2 \theta + (k_2(\Gamma) - 1)(\Gamma + 1)(3 \cos^2 \theta - 1)] + \epsilon^2 \frac{27}{16} \frac{(3\Gamma^2 - 14\Gamma + 3)(\Gamma - 1)}{(5 - 3\Gamma)^3},$$

where $k_2(\Gamma) = x_{0s}^3 \frac{1}{\hat{a}^2} \frac{dg_{22}}{dx} \Big|_{x=x_{0s}}$ is computed numerically in BP (see Table 1 there), $k_2(4/3) = 0.861$. The shape of the sonic surface is also shown on Fig. 4 for $\Gamma = 4/3$ and $a = \mathcal{M}$.

5 Conclusions

Using stream (Grad–Shafranov) equation (14) we have considered the problem of hydrodynamic accretion onto a resting Kerr black hole. The fact that in astrophysical situations, when this type of accretion could be realized (accretion of interstellar medium onto an isolated black hole), the temperature of the surrounding gas is nonrelativistic introduces small parameter in the problem and allows to obtain analytical solution of the stream equation (14) in the case of an arbitrary value of the rotational parameter a , extending thus the results of the work BP onto rapid rotation of the black hole.

Form of the sonic surface and corrections to the particle density are determined. However, because the sonic surface is located far from the horizon of a black hole these corrections are very small and flow picture does not differ drastically from the case of the accretion onto a Schwarzschild black hole save the dragging into rotation around Kerr black hole with Lense–Thirring angular velocity. Therefore, the related astrophysics (emergent radiation) does not change significantly from previous consideration by Shapiro, 1974. Presence of a magnetic field or small angular momentum of the gas at infinity relative to the black hole have much more influence on the accretion pattern and emergent radiation. So, the results of the present work lie primarily in the scope of mathematical physics and show that shock-free smooth solution exists in the problem of the steady state hydrodynamic accretion onto a rotating black hole with arbitrary $a \leq \mathcal{M}$.

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Figure Captions

- Figure 1.** Functions $y_0(x)$ corresponding to accretion and ejection solutions of equation (36) for $\Gamma = 4/3$.
Figure 2. Dependencies of y_1 on x for different values of Γ .
Figure 3. Function $G(x)$ for different values of Γ .
Figure 4. Stream lines and sonic surface for $\epsilon = 0.3$, $a = \mathcal{M}$ and $\Gamma = 4/3$.